11.3 Arithmetic progressions

At IGCSE level you learnt that a number sequence is an ordered set of numbers that satisfy a rule and that the numbers in the sequence are called the terms of the sequence. A number sequence is also called a progression.

The sequence 5, 8, 11, 14, 17, ... is called an arithmetic progression. Each term differs from the term before by a constant. This constant is called the common difference.

The notation used for arithmetic progressions is:

a = first term d = common difference l = last term

The first five terms of an arithmetic progression whose first term is a and whose common difference is d are:

a	$a+d$	$a+2d$	$a+3d$	$a+4d$
term 1	term 2	term 3	term 4	term 5

This leads to the formula:

$$n\text{th term} = a + (n-1)d$$

WORKED EXAMPLE 6

Find the number of terms in the arithmetic progression $-17, -14, -11, -8, \ldots, 58$.

Answers

nth term $= a + (n-1)d$ use $a = -17$, $d = 3$ and nth term $= 58$

$58 = -17 + 3(n-1)$ solve

$n - 1 = 25$

$n = 26$

WORKED EXAMPLE 7

The fifth term of an arithmetic progression is 4.4 and the ninth term is 7.6. Find the first term and the common difference.

Answers

fifth term = 4.4 $\Rightarrow a + 4d = 4.4$ -------(1)

ninth term = 7.6 $\Rightarrow a + 8d = 7.6$ -------(2)

(2) − (1), gives $4d = 3.2$

$d = 0.8$

Substituting in (1) gives $a + 3.2 = 4.4$

$a = 1.2$

First term = 1.2, common difference = 0.8.

WORKED EXAMPLE 8

The nth term of an arithmetic progression is $11 - 3n$. Find the first term and the common difference.

Answers

First term $= 11 - 3(1) = 8$ substitute $n = 1$ into nth term $= 11 - 3n$

second term $= 11 - 3(2) = 5$ substitute $n = 2$ into nth term $= 11 - 3n$

Common difference = second term − first term = −3.

The sum of an arithmetic progression

When the terms in a sequence are added together the resulting sum is called a series.

CLASS DISCUSSION

$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100 = ?$

It is said that at the age of eight, the famous mathematician Carl Gauss was asked to find the sum of the numbers from 1 to 100. His teacher expected this task to keep him occupied for some time but Gauss surprised his teacher by writing down the correct answer after just a couple of seconds. His method involved adding the numbers in pairs: $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, ...

1 Can you complete his method to find the answer?

2 Use Gauss's method to find the sum of

a $2 + 4 + 6 + 8 + \ldots + 394 + 396 + 398 + 400$

b $3 + 6 + 9 + 12 + \ldots + 441 + 444 + 447 + 450$

c $17 + 24 + 31 + 38 + \ldots + 339 + 346 + 353 + 360$.

3 Use Gauss's method to find an expression, in terms of n, for the sum

$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$.

It can be shown that the sum of an arithmetic progression, S_n, can be written as:

$$S_n = \frac{n}{2}(a+l) \quad \text{or} \quad S_n = \frac{n}{2}[2a+(n-1)d]$$

Proof:

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l$$

Reversing:

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a$$

Adding:

$$2S_n = n(a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l)$$

$$2S_n = n(a+l)$$

$$S_n = \frac{n}{2}(a+l)$$

Using $l = a + (n-1)d$, gives $S_n = \frac{n}{2}[2a + (n-1)d]$

It is useful to remember the following rule that applies for all progressions:

$$n\text{th term} = S_n - S_{n-1}$$

WORKED EXAMPLE 9

In an arithmetic progression, the first term is 25, the 19th term is –38 and the last term is –87. Find the sum of all the terms in the progression.

Answers

nth term $= a + (n-1)d$ — use nth term $= -38$ when $n = 19$ and $a = 25$

$-38 = 25 + 18d$ — solve

$d = -3.5$

nth term $= a + (n-1)d$ — use nth term $= -87$ when $a = 25$ and $d = -3.5$

$-87 = 25 - 3.5(n-1)$ — solve

$n - 1 = 32$

$n = 33$

$S_n = \frac{n}{2}(a + l)$ — use $a = 25$, $l = -87$ and $n = 33$

$S_{33} = \frac{33}{2}(25 - 87)$

$= -1023$

WORKED EXAMPLE 10

The 12th term in an arithmetic progression is 8 and the sum of the first 13 terms is 78. Find the first term of the progression and the common difference.

Answers

nth term $= a + (n-1)d$ — use nth term $= 8$ when $n = 12$

$8 = a + 11d$ ---------- (1)

$S_n = \frac{n}{2}[2a + (n-1)d]$ — use $n = 13$ and $S_{13} = 78$

$78 = \frac{13}{2}(2a + 12d)$ — simplify

$6 = a + 6d$ ---------- (2)

(1) – (2) gives $5d = 2$

$d = 0.4$

Substituting $d = 0.4$ in equation (1) gives $a = 3.6$.

First term = 3.6, common difference = 0.4.

WORKED EXAMPLE 11

The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = 5n^2 - 3n$.

a Find the first term and the common difference.

b Find an expression for the nth term.

Answers

a $S_1 = 5(1)^2 - 3(1) = 2 \Rightarrow$ first term = 2

$S_2 = 5(2)^2 - 3(2) = 14 \Rightarrow$ first term + second term = 14

second term = 14 − 2 = 12

First term = 2, common difference = 10.

b Method 1:

nth term $= a + (n-1)d$ use $a = 2$, $d = 10$

$= 2 + 10(n-1)$

$= 10n - 8$

Method 2:

nth term $= S_n - S_{n-1} = 5n^2 - 3n - [5(n-1)^2 - 3(n-1)]$

$= 5n^2 - 3n - (5n^2 - 10n + 5 - 3n + 3)$

$= 10n - 8$

Exercise 11.3

1 The first term in an arithmetic progression is a and the common difference is d.

Write down expressions, in terms of a and d, for the fifth term and the 14th term.

2 Find the sum of each of these arithmetic series.

a $2 + 9 + 16 + \ldots$ (15 terms)

b $20 + 11 + 2 + \ldots$ (20 terms)

c $8.5 + 10 + 11.5 + \ldots$ (30 terms)

d $-2x - 5x - 8x - \ldots$ (40 terms)

3 Find the number of terms and the sum of each of these arithmetic series.

a $23 + 27 + 31 \ldots + 159$

b $28 + 11 - 6 - \ldots -210$

4 The first term of an arithmetic progression is 2 and the sum of the first 12 terms is 618.

Find the common difference.

5 In an arithmetic progression, the first term is −13, the 20th term is 82 and the last term is 112.

a Find the common difference and the number of terms.

b Find the sum of the terms in this progression.

6 The first two terms in an arithmetic progression are 57 and 46. The last term is −207. Find the sum of all the terms in this progression.

7 The first two terms in an arithmetic progression are –2 and 5. The last term in the progression is the only number in the progression that is greater than 200. Find the sum of all the terms in the progression.

8 The first term of an arithmetic progression is 8 and the last term is 34. The sum of the first six terms is 58. Find the number of terms in this progression.

9 Find the sum of all the integers between 100 and 400 that are multiples of 6.

10 The first term of an arithmetic progression is 7 and the eleventh term is 32. The sum of all the terms in the progression is 2790. Find the number of terms in the progression.

11 Rafiu buys a boat for $15 500. He pays for this boat by making monthly payments that are in arithmetic progression. The first payment that he makes is $140 and the debt is fully repaid after 31 payments. Find the fifth payment.

12 The eighth term of an arithmetic progression is –10 and the sum of the first twenty terms is –350.

a Find the first term and the common difference.

b Given that the nth term of this progression is –97, find the value of n.

13 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = 4n^2 + 2n$. Find the first term and the common difference.

14 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = -3n^2 - 2n$. Find the first term and the common difference.

15 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = \frac{n}{12}(4n+5)$. Find an expression for the nth term.

16 A circle is divided into twelve sectors. The sizes of the angles of the sectors are in arithmetic progression. The angle of the largest sector is 6.5 times the angle of the smallest sector. Find the angle of the smallest sector.

17 An arithmetic sequence has first term a and common difference d. The sum of the first 25 terms is 15 times the sum of the first 4 terms.

a Find a in terms of d.

b Find the 55th term in terms of a.

18 The eighth term in an arithmetic progression is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms.

CHALLENGE Q

19 The first term of an arithmetic progression is $\cos^2 x$ and the second term is 1.

a Write down an expression, in terms of $\cos x$, for the seventh term of this progression.

b Show that the sum of the first twenty terms of this progression is $20 + 170 \sin^2 x$.

$\frac{1}{10} \quad \frac{1}{100} + \frac{1}{2000}$

$\frac{}{000}$

$15x^3 + 90x - \frac{270}{x} + \frac{405}{x^3}$

$3x^3 + \frac{15}{4}x^4 + \frac{5}{2} + \frac{15}{16x^4}$

$\frac{}{^3} + \frac{1}{64x^{12}}$

b $175\,000x^3$

d $720x^3$

f $-5376x^3$

$50\,000x^3$

$1\,204\,160\,000x^3$

$0x + 45x^2$

$6x + 112x^2$

$21x + 189x^2$

$+ 2916x + 4860x^2$

$3 - 59049x + 78732x^2$

$+ 512x + 448x^2$

$3125 - 3\,515\,625x^2$

$12\,500x^4$

$3576x^{10} - 13\,107\,200x^9y$

$728\,000x^8y^2$

$2x + 60x^2 + 160x^3$

10 -945

11 $\frac{21}{2}$

12 $a = \frac{3}{n-2}$

Exercise 11.3

1 $a + 4d,\ a + 13d$

2 a 765 b -1310

c 907.5 d $-2420x$

3 a 35, 3185

b 15, -1365

4 9

5 a 5, 26 b 1287

6 -1875

7 2985

8 40

9 12 450

10 45

11 \$236

12 a 11, -3 b 37

13 6, 8

14 $-5, -6$

15 $\frac{1}{12}(8n + 1)$

16 $8°$

17 a $a = 6d$

b $10a$

19 a $6 - 5\cos^2 x$

20 b 901