CHALLENGE Q

20 The sum of the digits in the number 56 is 11. ($5 + 6 = 11$)

a Show that the sum of the digits of the integers from 15 to 18 is 30.

b Find the sum of the digits of the integers from 1 to 100.

11.4 Geometric progressions

The sequence 7, 14, 28, 56, 112, ... is called ageometric progression. Each term is double the preceding term. The constant multiple is called the common ratio.

Other examples of geometric progressions are:

Progression	Common ratio
$1, -2, 4, -8, 16, -32, \ldots$	-2
$81, 54, 36, 24, 16, 10\frac{2}{3}, \ldots$	$\frac{2}{3}$
$-8, 4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$	$-\frac{1}{2}$

The notation used for a geometric progression is:

a = first term $\qquad$ r = common ratio

The first five terms of a geometric progression whose first term is a and whose common ratio is r are:

a	ar	ar^2	ar^3	ar^4
term 1	term 2	term 3	term 4	term 5

This leads to the formula:

$$n\text{th term} = ar^{n-1}$$

WORKED EXAMPLE 12

The third term of a geometric progression is 144 and the common ratio is $\frac{3}{2}$. Find the seventh term and an expression for the nth term.

Answers

nth term $= ar^{n-1}$ $\qquad$ use nth term = 144 when $n = 3$ and $r = \frac{3}{2}$

$$144 = a\left(\frac{3}{2}\right)^2$$

$$a = 64$$

$$\text{seventh term} = 64\left(\frac{3}{2}\right)^6 = 729$$

$$n\text{th term} = ar^{n-1} = 64\left(\frac{3}{2}\right)^{n-1}$$

WORKED EXAMPLE 13

The second and fourth terms in a geometric progression are 108 and 48 respectively. Given that all the terms are positive, find the first term and the common ratio. Hence, write down an expression for the nth term.

Answers

$108 = ar$ ...... (1) $\quad 48 = ar^3$ ....(2)

(2)÷ (1) gives $\frac{ar^3}{ar} = \frac{48}{108}$

$r^2 = \frac{4}{9}$

$r = \pm\frac{2}{3}$ all terms are positive $\Rightarrow r > 0$

$r = \frac{2}{3}$

Substituting $r = \frac{2}{3}$ into equation (1) gives $a = 162$.

First term = 162, common ratio = $\frac{2}{3}$, nth term = $162\left(\frac{2}{3}\right)^{n-1}$.

WORKED EXAMPLE 14

The nth term of a geometric progression is $30\left(-\frac{1}{2}\right)^n$. Find the first term and the common ratio.

Answers

first term = $30\left(-\frac{1}{2}\right)^1 = -15$

second term = $30\left(-\frac{1}{2}\right)^2 = 7.5$

Common ratio = $\frac{\text{2nd term}}{\text{1st term}} = \frac{7.5}{-15} = -\frac{1}{2}$

First term = −15, common ratio = $-\frac{1}{2}$.

WORKED EXAMPLE 15

In the geometric sequence 2, 6, 18, 54, ... which is the first term to exceed 1000000?

Answers

$n\text{th term} = ar^{n-1}$	use $a = 2$ and $r = 3$
$2 \times 3^{n-1} > 1000000$	divide by 2 and take logs
$\log_{10} 3^{n-1} > \log_{10} 500000$	use the power rule for logs
$(n-1)\log_{10} 3 > \log_{10} 500000$	divide both sides by $\log_{10} 3$

$$n - 1 > \frac{\log_{10} 500000}{\log_{10} 3}$$

$$n - 1 > 11.94...$$

$$n > 12.94...$$

The 13th term is the first to exceed 1000000.

CLASS DISCUSSION

In this class discussion you are not allowed to use a calculator.

1 Consider the sum of the first 10 terms, S_{10}, of a geometric progression with $a = 1$ and $r = 5$.

$S_{10} = 1 + 5 + 5^2 + 5^3 + ... + 5^7 + 5^8 + 5^9$

a Multiply both sides of the equation above by the common ratio, 5, and complete the following statement.

$5S_{10} = 5 + 5^2 + 5^{\cdots} + 5^{\cdots} + ... + 5^{\cdots} + 5^{\cdots} + 5^{\cdots}$

b What happens when you subtract the equation for S_{10} from the equation for $5S_{10}$?

c Can you find an alternative way of expressing the sum S_{10}?

2 Use the method from **question 1** to find an alternative way of expressing each of the following

a $3 + 3 \times 2 + 3 \times 2^2 + 3 \times 2^3 + ...$ (12 terms)

b $32 + 32 \times \frac{1}{2} + 32 \times \left(\frac{1}{2}\right)^2 + 32 \times \left(\frac{1}{2}\right)^3 + ...$ (15 terms)

c $27 - 18 + 12 - 8 + ...$ (20 terms)

It can be shown that the sum of a geometric progression, S_n, can be written as:

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r - 1}$$

Note:
For these formulae, $r \neq 1$

Either formula can be used but it is usually easier to

- use the first formula when $-1 < r < 1$.
- use the second formula when $r > 1$ or when $r < -1$.

Proof:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \quad \text{---------- (1)}$$

$$r \times (1): \quad rS_n = ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n \quad \text{---------- (2)}$$

$$(2) - (1): \quad rS_n - S_n = ar^n - a$$

$$(r-1)S_n = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Multiplying numerator and denominator by −1 gives the alternative formula $S_n = \frac{a(1 - r^n)}{1 - r}$.

WORKED EXAMPLE 16

Find the sum of the first ten terms of the geometric series 2 + 6 + 18 + 54 + …

Answers

$S_n = \frac{a(r^n - 1)}{r - 1}$ use $a = 2$, $r = 3$ and $n = 10$

$S_{12} = \frac{2(3^{10} - 1)}{3 - 1}$ simplify

$= 59\,048$

WORKED EXAMPLE 17

The second term of a geometric progression is 9 less than the first term. The sum of the second and third terms is 30. Given that all the terms in the progression are positive, find the first term.

Answers

second term = first term −9

$ar = a - 9$ rearrange to make a the subject

$a = \frac{9}{1 - r}$ -------(1)

second term + third term = 30

$ar + ar^2 = 30$ factorise

$ar(1 + r) = 30$ -------(2)

(2) ÷ (1) gives $\frac{ar(1 + r)}{a} = \frac{30(1 - r)}{9}$ simplify

$3r^2 + 13r - 10 = 0$ factorise and solve

$(3r - 2)(r + 5) = 0$

$r = \frac{2}{3}$ or $r = -5$ all terms are positive $\Rightarrow r > 0$

$r = \frac{2}{3}$

Substituting $r = \frac{2}{3}$ into (1) gives $a = 27$.

First term is 27.

Exercise 11.4

1 Identify whether the following sequences are geometric. If they are geometric, write down the common ratio and the eighth term.

a 1, 2, 4, 6, ...

b −1, 4, −16, 64, ...

c 81, 27, 9, 3, ...

d $\frac{2}{11}, \frac{3}{11}, \frac{5}{11}, \frac{8}{11}, \ldots$

e 2, 0.4, 0.08, 0.16, ...

f −5, 5, −5, 5, ...

2 The first term in a geometric progression is a and the common ratio is r. Write down expressions, in terms of a and r, for the ninth term and the 20th term.

3 The third term of a geometric progression is 108 and the sixth term is −32. Find the common ratio and the first term.

4 The first term of a geometric progression is 75 and the third term is 27. Find the two possible values for the fourth term.

5 The second term of a geometric progression is 12 and the fourth term is 27. Given that all the terms are positive, find the common ratio and the first term.

6 The sixth and 13th terms of a geometric progression are $\frac{5}{2}$ and 320 respectively. Find the common ratio, the first term and the 10th term of this progression.

7 The sum of the second and third terms in a geometric progression is 30. The second term is 9 less than the first term. Given that all the terms in the progression are positive, find the first term.

8 Three consecutive terms of a geometric progression are x, $x+6$ and $x+9$. Find the value of x.

9 In the geometric sequence $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$ which is the first term to exceed 500000?

10 In the geometric sequence 256, 128, 64, 32, ... which is the first term that is less than 0.001?

11 Find the sum of the first eight terms of each of these geometric series.

a $4 + 8 + 16 + 32 + \ldots$

b $729 + 243 + 81 + 27 + \ldots$

c $2 - 6 + 18 - 54 + \ldots$

d $-5000 + 1000 - 200 + 40 - \ldots \ldots$

12 The first four terms of a geometric progression are 1, 3, 9 and 27. Find the smallest number of terms that will give a sum greater than 2000000.

13 A ball is thrown vertically upwards from the ground. The ball rises to a height of 10m and then falls and bounces. After each bounce it rises to $\frac{4}{5}$ of the height of the previous bounce.

a Write down an expression, in terms of n, for the height that the ball rises after the nth impact with the ground.